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On the Closed Form Solution of Troesch's Problem

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The closed form solution of Troesch's problem is developed in terms of Jacobian elliptic functions. From the closed form solution two interesting properties of Troesch's problem can be found. The first is the location of a pole, the second is branching or bifurcation behavior. Depending on the value of the parameter n, the problem possesses a continuous solution and possibly one or more discontinuous solutions. Numerical evaluation of the closed form solution as well as numerical integration is carried out for $n = 5(1)10$ to show the effects of the discontinuous solutions on the numerical computations.

1. INTRODUCTION

Troesch [13] studied the two-point boundary value problem

$$\ddot{y} = n \sinh ny \tag{1.1}$$

$$y(0) = 0, \qquad y(1) = 1 \tag{1.2}$$

to explain the difficulties encountered in the numerical solution of a two-point boundary value problem for a system of nonlinear ordinary differential equations which occurred in an investigation of the confinement of a plasma column by radiation pressure. He showed that the initial value problem associated with (1.1) has a pole approximately at

$$t_\infty = (1/n) \ln(8/\dot{y}(0)) \tag{1.3}$$

which makes the solution of (1.1), (1.2) by shooting methods difficult, the difficulty increasing with increasing n.

In addition to its intrinsic interest, Troesch's problem has become something of a test case for methods of solving unstable two-point boundary value problems because of its difficulties. Tsuda, Ichida, and Kiyono [14] solved (1.1), (1.2) by a Monte Carlo path integral calculation for $n \leqslant 5$. Roberts and Shipman [10] used a combination of methods to solve (1.1), (1.2) for $n = 5$, 6 and 10. Miele, Aggarwal, and Tietze [9] obtained solutions accurate to six significant figures by a combination of the multipoint method and modified quasilinearization. They also found that with the appropriate initial profile, modified quasilinearization alone gives accurate results. Jones [7] solved (1.1), (1.2) by using Gear's integration method for nonstiff equations and modifying the correction to $\dot{y}(0)$ so that the calculated value of $| y^{(m)}(1) - 1 |$ decreases as m, the iteration count number, increases. Chiou and Na [4] applied the method of transformation groups to convert the original two-point boundary value problem to an initial value problem which can be solved noniteratively. However in the course of the transformation n is transformed to an $\bar{n}$ whose value is not known, so that the initial value problem must be solved for several values of $\bar{n}$ and the solution for that value of $\bar{n}$ corresponding to the given n is found by interpolation. Kubiček and Hlaváček [8] have also obtained values of $\dot{y}(0)$ for irregularly spaced values of n up to 29.71 by the so-called parameter mapping technique and have also given a transformation of Troesch's problem which exchanges the independent and dependent variables, and thus eliminates the pole. It should be noted that most of the methods cited with the exception of Roberts and Shipman [10] rely on some particular property of the structure of Troesch's problem or knowledge of the general shape of its solution.

It should also be noted that Henrici [6] solved the closely related problem

$$\ddot{y} = \sinh(y) - 2 \tag{1.4}$$

$$y(0) = y(1) = 0 \tag{1.5}$$

by a finite-difference method without encountering any difficulties. However, we performed some preliminary numerical experiments which indicated that Troesch's problem poses difficulties for finite difference methods too. Indeed, the bound for the discretization error e_n as given by Henrici [6] for a fourth-order method ($p = 4$) with a mesh size of 0.1 ($h = 0.1$) is 10^{-7} for $n = 1$ but is 10^{-1} for $n = 5$. It appears that nonuniform meshes are necessary to obtain satisfactory accuracy when finite-difference methods are applied to Troesch's problem.

In his lectures on the multipoint methods, Bulirsch [2] remarked (but did not exhibit) that (1.1), (1.2) has a closed form solution in terms of elliptic functions. In their book [12] Stoer and Bulirsch give without the derivation the closed form solution.

In this paper we give one form of the analytical solution of (1.1), (1.2). Since it can be used to generate accurate values of $y(t)$ for the interval $0 \leqslant t \leqslant 1$, it should enhance the usefulness of (1.1), (1.2) as a test problem. We also note some properties of the closed form of the solution $y(t)$ and exploit these to help explain phenomena observed in the numerical solution of (1.1). In addition we point out that the solutions of (1.1), (1.2) exhibit a form of branching or bifurcation and we discuss the nature and multiplicity of the solutions for $n = 1, 2, 3, \ldots, 10$.

2. Closed Form Solution

The closed form solution of (1.1), (1.2) may be obtained formally by first multiplying both sides of (1.1) by $\dot{y}$ and integrating to obtain the familiar first integral

$$\dot{y}^2 = 2\cosh(ny) + C. \tag{2.1}$$

The initial condition in (1.2) is used to evaluate the constant C

$$C = \dot{y}^2(0) - 2. \tag{2.2}$$

The implicit solution of (1.1), (1.2) is then given by

$$t = \int_0^y (dv/(C + 2\cosh nv)^{1/2}). \tag{2.3}$$

If we replace $2\cosh nv$ by $4\sinh^2(nv/2) + 2$ and if we make a change of variable $z = inv/2$, then (2.3) appears as

$$t = \frac{-2i}{n(2 + C)^{1/2}} \int_0^{iny/2} \frac{dz}{(1 - (4/(2 + C))\sin^2 z)^{1/2}} \tag{2.4}$$

or

$$\frac{in(2 + C)^{1/2}\, t}{2} = \int_0^{iny/2} \frac{dz}{(1 - (4/(2 + C))\sin^2 z)^{1/2}}\,. \tag{2.5}$$

Now (2.5) is of the form of [1, (16.1.3)]; namely,

$$u = \int_0^{\varphi} \frac{d\theta}{(1 - m\sin^2\theta)^{1/2}} \tag{2.6}$$

where

$$u = (in(2 + C)^{1/2}/2)\, t, \tag{2.7}$$

$$\varphi = iny/2, \tag{2.8}$$

$$m = 4/(2 + C). \tag{2.9}$$

Now in terms of the Jacobian elliptic function $\operatorname{sn}(u \mid m)$ we have the relationship $\sin \varphi = \operatorname{sn}(u \mid m)$[1, (16.1.5)], so we may write

$$\sin \frac{iny}{2} = \operatorname{sn}\left(\frac{in(2+C)^{1/2}}{2} t \,\middle|\, \frac{4}{2+C}\right). \tag{2.10}$$

Since $\sin iz = i \sinh z$, $\operatorname{sn}(iu \mid m) = i \operatorname{sc}(u \mid m_1)$, where $m_1 = 1 - m$ (Jacobi's imaginary transformation, [1, (16.20.1)]), the closed form solution of (1.1), (1.2) is given by

$$y(t) = \frac{2}{n} \sinh^{-1} \left\{ \operatorname{sc}\left(\frac{n(2+C)^{1/2}}{2} t \,\middle|\, 1 - \frac{4}{2+C}\right)\right\}. \tag{2.11}$$

However, the parameter $m = 1 - (4/(2 + C))$ in the cases of interest to us is negative. We have found it more convenient to express (2.11) in a form in which the parameter is positive and the dependence of $y(t)$ and $\dot{y}(t)$ on this parameter is explicitly exhibited. This can be done by means of the change of parameter given in [1, (16.10.1)]; namely, $\mu = m/(1 + m)$, $\mu_1 = 1/(1 + m)$, $v = u/(\mu_1)^{1/2}$ (where since $m > 0$ in the formulas, we take $m = -[1 - 4/(2 + C)]$), and by means of [1, (16.10.2), (16.10.3)], and the boundary conditions (1.2) and (2.2). Further it is often necessary to compute the first derivative $\dot{y}(t)$(for example, to evaluate the first integral of (2.1)), whose expression can of course be obtained by differentiating the closed form solution for $y(t)$ and using [1, (16.16.9)]. The resulting expressions for the closed form solution of (1.1) and its derivative are then

$$y(t) = \frac{2}{n} \sinh^{-1} \left\{ \frac{\dot{y}(0)}{2} \operatorname{sc}\left(nt \,\middle|\, 1 - \frac{1}{4} \dot{y}^2(0)\right)\right\} \tag{2.12}$$

$$\dot{y}(t) = \frac{\dot{y}(0)\{\operatorname{dc}(nt \mid 1 - (1/4)\, \dot{y}^2(0)) \operatorname{nc}(nt \mid 1 - (1/4)\, \dot{y}^2(0))\}}{\cosh(n/2)\, y(t)}. \tag{2.13}$$[1]

That (2.12) satisfies (1.1) can of course be verified directly. Further, $y(0) = 0$ since $\operatorname{sc}(0) = \operatorname{sn}(0)/\operatorname{cn}(0) = 0/1 = 0$. The boundary condition $y(1) = 1$ will be satisfied if m satisfies the transcendental equation

$$\sinh(n/2)/(1 - m)^{1/2} = \operatorname{sc}(n \mid m) \tag{2.14}$$

where $m = 1 - \frac{1}{4}\dot{y}^2(0)$ and the required value of $\dot{y}(0)$ is equal to $2\,(1 - m)^{1/2}$. It may be objected that (2.12) is not quite the "closed-form" solution of (1.1), (1.2) since $\dot{y}(0)$ is only defined implicitly by (2.14). However (2.14) lends itself to a simple

[1] One of the referees pointed out that if $\cosh(n/2)y(t)$ is expressed in terms of (2.12), and if [1, (16.9.3)] is used with $m = 1 - \frac{1}{4}\dot{y}^2(0)$ (and therefore $m_1 = \frac{1}{4}\dot{y}^2(0)$), then (2.13) simplifies to $\dot{y}(t) = \dot{y}(0) \operatorname{nc}(nt \mid 1 - \frac{1}{4}\dot{y}^2(0))$.

graphical analysis, from which important properties of the solution of (1.1), (1.2) can be deduced.

We have given the details of the closed form solution of Troesch's problem in the form (1.1), (1.2) because this is the form treated in the cited references. We observe that it is possible to obtain a closed form solution to the most general problem of the form (2.1), (2.2):

$$\ddot{y} = a \sinh by \qquad a, b > 0,\ t_0 \leqslant t \leqslant t_f, \tag{2.15}$$

$$y(t_0) = A, \qquad y(t_f) = B. \tag{2.16}$$

Little essentially new emerges in this more general treatment except for some complications in satisfying the boundary conditions. In the first place, by a linear transformation of the independent variable, (2.15), (2.16) can be transformed to a problem of the same form with $t_0 = 0$, $t_f = 1$. Next, by scaling the dependent variable, (2.15), (2.16) can be transformed into a problem of the same form with $a = b = n$. Finally, from (2.11) we can derive the expression

$$y(t; c_1, c_2) = (2/n) \sinh^{-1} \{\mathrm{sc}(nc_1(t + c_2) | 1 - (1/c_1^2))\} \tag{2.17}$$

which satisfies (2.15) (with $a = b = n$). The parameters c_1 and c_2 are then determined by solving (if possible) the pair of simultaneous transcendental equations of the form

$$\sinh(nA/2) = \mathrm{sc}(nc_1c_2 \mid 1 - (1/c_1^2)) \tag{2.18}$$

$$\sinh(nB/2) = \mathrm{sn}(nc_1(c_2 + 1) | 1 - (1/c_1^2)). \tag{2.19}$$

We have not carried out the analysis of (2.18), (2.19), which does not appear to yield insights into the behavior of the solutions of (2.15), (2.16) as readily as (2.12) does with respect to (1.1), (1.2).

3. Properties of the Solution of Troesch's Problem

Troesch's problem has two interesting (and related) properties which bear on the numerical difficulties it presents. These properties are easily derived from the closed form solution. In the first place, the solution has a pole located approximately at

$$t_\infty = (1/n) \ln(8/\dot{y}(0)). \tag{3.1}$$

Troesch [5, 13] derived this expression for t_∞ from a consideration of the first integral (2.3). Expression (3.1) for t_∞ can also be obtained, and in a very simple manner, from the closed form expression as follows.

From (2.12) we recognize that a pole of $y(t)$ occurs at a pole of $\mathrm{sc}(nt \mid 1 - \frac{1}{4}\dot{y}^2(0))$. By [1, Tables 16.2 and 16.5] we see that $\mathrm{sc}(u \mid m)$ has a pole at $u = K$, where K, the complete elliptic integral of the first kind, is defined as

$$K(m) = \int_0^{\pi/2} (d\theta/(1 - m \sin^2 \theta)^{1/2}) \tag{3.2}$$

By [1, (17.3.26)] for $m \approx 1$

$$K(m) \approx (1/2) \ln(16/(1 - m)) \tag{3.3}$$

so the pole of $\mathrm{sc}(nt \mid 1 - \frac{1}{4}\dot{y}^2(0))$ occurs at $nt = K(m)$ or

$$nt \approx \frac{1}{2} \ln \frac{16}{1 - m} = \frac{1}{2} \ln \frac{16}{(1/4)\,\dot{y}^2(0)} \tag{3.4}$$

or the pole is given by

$$t_\infty \approx (1/n) \ln(8/\dot{y}(0)) \tag{3.5}$$

in agreement with (3.1). The significance of (3.1) for shooting methods, as has been noted frequently, is that reasonable guesses for the missing initial condition $\dot{y}(0)$ may give a numerical solution to (1.1) which has a pole for $0 \leqslant t \leqslant 1$. Stoer and Bulirsch [12] give a similar derivation of (3.1).

The second property of the solutions of (1.1), (1.2) we wish to point out is a kind of branching or bifurcation which they exhibit. This behavior might be expected, since the similar problem

$$\ddot{y} + \kappa \sin y = 0, \tag{3.6}$$

$$\dot{y}(0) = \dot{y}(1) = 0, \tag{3.7}$$

which arises in the study of the buckling of a thin rod under compression, is used by Stakgold [11] as a model problem to illustrate the branching phenomenon.

When the boundary condition $y(1) = 1$ is introduced into (2.12), we obtain the transcendental equation (2.14), which is the relationship determining m as a function of n, and which, as was mentioned earlier, lends itself to a simple graphical analysis. If the left- and right-hand sides of (2.14) are plotted as a function of m, the intersections of the two curves are the roots of (2.14). The roots in turn give the missing initial condition $\dot{y}(0)$ through the relationship $m = 1 - \frac{1}{4}\dot{y}^2(0)$. Now the left-hand side of (2.14), $\sinh(n/2)/(1 - m)^{1/2}$, increases monotonically from $\sinh(n/2)$ to ∞ as m goes from 0 to 1. The general shape of the right-hand side of (2.14), $\mathrm{sc}(n \mid m)$, can be deduced from the location of its zeros and poles, which are, respectively, at $0, 2K, 4K, \ldots$, and $K, 3K, 5K, \ldots$. Since K is related to m through (3.2), for a given value of n, $\mathrm{sc}(n \mid m)$ will have a zero (pole) if a value of m, $0 \leqslant m \leqslant 1$, can be found such that $K(m) = n/2, n/4, n/6, \ldots, (K(m) = n, n/3, n/5, \ldots)$. And since $\mathrm{sc}(n \mid 0) = \tan n$, the number of zeroes (poles) will increase by one as n increases through even (odd) multiples of $\pi/2$. With this information, graphs such as the ones in Fig. 1 can be sketched. From this informal graphical

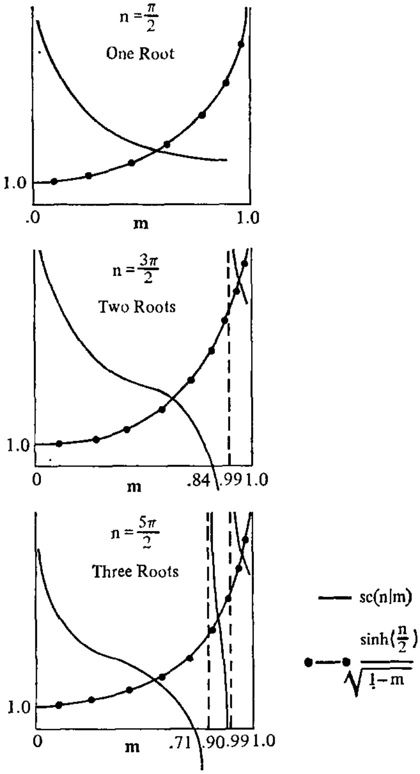

FIG. 1. Sketches of sc(n | m) and $\sinh(n/2)/(1 - m)^{1/2}$ versus m.

analysis it is apparent that (2.14) has $k + 1$ roots for $(2k + 1)\pi/2 \leqslant n < (2k + 3)\pi/2$. It would thus appear that as n increases, branching or bifurcation of the solutions of (1.1), (1.2) occurs at odd multiples of $\pi/2$. However, in contrast to Stakgold's [11] model problem, only the solution corresponding to the largest value of m, $m_{\max}$, remains finite for $0 \leqslant t \leqslant 1$, the remaining "solutions" all having a pole at some $\bar{t}_\infty$, $0 \leqslant \bar{t}_\infty \leqslant 1$. We call the latter "discontinuous solutions." They have the property that they meet the boundary conditions (1.2) and satisfy the differential equation (1.1) everywhere in [0, 1] with the exception of a finite number of points. Although the discontinuous solutions may not have physical meaning, they seem to have a bearing on the numerical behavior of the solution of (1.1), (1.2), discussed in the next section. Continuous and discontinuous solutions for $n = 5$ and 10 are shown in Figs. 2 and 3, respectively.

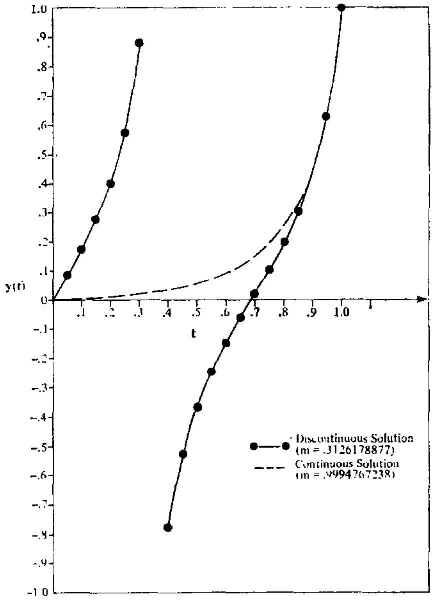

FIG. 2. Closed form solutions for $n = 5$.

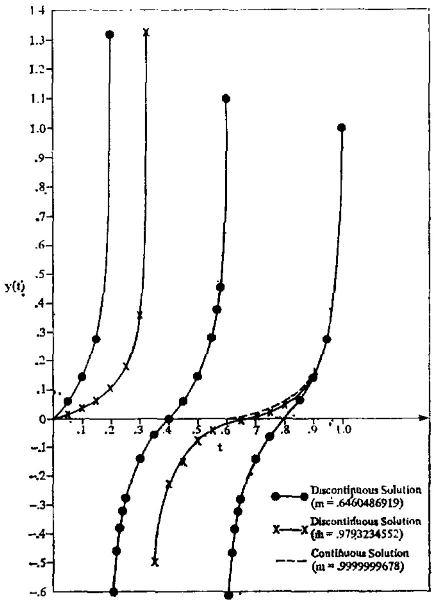

FIG. 3. Closed form solutions for $n = 10$.

4. Numerical Behavior of the Solutions

We remark first that our computation of the solution of (1.1), (1.2) from the closed form (2.12), (2.13) is straightforward once m (and then $\dot{y}(0)$) has been found from (2.14). In the following all the formulas cited for the various Jacobian elliptic functions may be found in [1]. The functions $\mathrm{sc}(u \mid m)$, $\mathrm{nc}(u \mid m)$, and $\mathrm{dc}(u \mid m)$ occurring in (2.12), (2.13) are defined in terms of the functions $\mathrm{sn}(u \mid m)$, $\mathrm{cn}(u \mid m)$, and $dn(u \mid m)$[1, (16.3.1)–(16.3.3)] which are evaluated from their respective series in powers of the nome $q = \exp(-\pi K'/K)$[1, (16.23.1)–(16.23.3)]. For larger values of m (in our work $m \geqslant 0.8$) Landen's decreasing transformation is employed to ensure that $q < 0.001$ [1, (16.12.1)–(16.12.4)] in which case only three or four terms of the series need be evaluated for an accuracy of 10^{-8}. The complete elliptic integral is evaluated from an approximation due to Hastings ((17.3.34)) which also has a maxium error of $2(10^{-8})$.

The difficulty in computing the solution to Troesch's problem from the closed form comes in solving (2.14) for m, since as n grows, the roots have a point of accumulation at $m = 1$. It becomes increasingly more difficult to extract the root

TABLE I

Roots and Poles of $\sinh(n/2)/(1 - m)^{1/2} = \mathrm{sc}(n \mid m)$

n	m	$\dot{y}(0)$	Error[a]	Comment
1	0.8214081054	0.8452026845	(10^{-9})	soln.
2	0.9327580075	0.5186212200	(10^{-9})	soln.
3	0.9836666214	0.2556042136	(10^{-9})	soln.
4	0.9968707071	0.1118801662	(10^{-9})	soln.
5	0.3126178877	1.658170211	(10^{-9})	discont. soln.
5	0.9994767238	0.04575046116	(10^{-9})	soln.
6	0.6698731471	1.149133330	(10^{-9})	discont. soln.
6	0.9999194408	0.01795094997	(10^{-9})	soln.
7	0.8379720769	0.8050538442	(10^{-8})	discont. soln.
7	0.9999882093	0.006867509416	(10^{-6})	soln.
8	0.08834209269	1.909615571	$2.8(10^{-9})$	discont. soln.
8	0.9192082346	0.5684778461	$2.8(10^{-9})$	discont. soln.
8	0.9999983266	0.002587169460	$7.3(10^{-6})$	soln.
9	0.4374757450	1.500032339	$1.2(10^{-9})$	discont. soln.
9	0.9592742677	0.4036123497	$2.7(10^{-9})$	discont. soln.
9	0.9999997669	0.0009655844857	$2.3(10^{-4})$	soln.
10	0.6460486919	1.189876141	$5.0(10^{-9})$	discont. soln.
10	0.9793234552	0.2875868197	$1.4(10^{-9})$	discont. soln.
10	0.9999999678	0.0003583377707	$1.2(10^{-2})$	soln.

[a] error $= \mid (\sinh(n/2)/(1 - m)^{1/2}) - \mathrm{sc}(n \mid m) \mid$.

corresponding to the continuous solution from the roots corresponding to the discontinuous solutions and increasingly more difficult to compute $m_{\max}$ accurately. In Table I are tabulated n, m, $\dot{y}(0)$, and the residual error $|(\sinh(n/2)/(1 - m)^{1/2}) - \mathrm{sc}(n \mid m)|$ for the roots of (2.14) (corresponding to continuous and discontinuous solutions) for $n = 1, 2, \ldots, 10$. Note that for $n = 8, 9, 10$ there are two discontinuous solutions. It will be observed that while the residual error is of the order of 10^{-9} for $n = 1, \ldots, 6$, it has increased to 10^{-6} for $n = 7$, to 10^{-4} for $n = 9$, and to 10^{-2} for $n = 10$. Again the graphs indicate the reason for this. As n increases the graph of $\mathrm{sc}(n \mid m)$ in the neighborhood of $m_{\max}$ becomes increasingly steep, so that small changes in m produce large changes in $\mathrm{sc}(n \mid m)$. The problem of solving (2.14) becomes more and more sensitive (ill-posed) with increasing n, and then the error in computing $m_{\max}$ also grows.

Since Troesch's problem is sensitive in the forward direction, if one were solving it by a numerical method that requires integration of (1.1), one might consider integrating it in the backward direction, that is from 1 to 0. It is our belief that here the existence of the discontinuous solutions make backward integration subject to the same difficulties as forward integration. From Table II, which gives the values

TABLE II

Closed Form Solution

n	$\dot{y}(0)$	$y(1)$	$\dot{y}(1)$	Comment
1	0.8452026845	1.000000000	1.341837966	soln.
2	0.5186212200	1.000000000	2.406939711	soln.
3	0.2556042136	1.000000000	4.266223175	soln.
4	0.1118801662	0.9999999999	7.254582910	soln.
5	1.658170211	0.9999999999	12.21349333	discont. soln.
5	0.04575046116	0.9999999999	12.10049568	soln.
6	1.149133330	0.9999999999	20.06867706	discont. soln.
6	0.01795094997	1.000000000	20.03575367	soln.
7	0.8050538442	0.9999999999	33.09505159	discont. soln.
7	0.006867509416	0.9999999999	33.08526669	soln.
8	1.909615571	1.000000000	54.61322834	discont. soln.
8	0.5684778461	1.000000000	54.58279367	discont. soln.
8	0.002587169460	1.000000000	54.57982377	soln.
9	1.500032339	0.9999999999	90.01851810	discont. soln.
9	0.4036123497	1.000000000	90.00692027	discont. soln.
9	0.0009655844857	0.9999999996	90.00618074	soln.
10	1.189876141	0.9999999999	148.4111963	discont. soln.
10	0.2875868197	0.9999999999	148.4067081	discont. soln.
10	0.0003583377707	1.000000012	148.4065422	soln.

of $y(1)$ and $\dot{y}(1)$ as computed from the closed form solution, it appears that for each n, $\dot{y}(1)$ for at least one discontinuous solution approaches $\dot{y}(1)$ for the continuous solution. Thus we conjectured that if backward integration is initiated with $\dot{y}(1)$ slightly different from the value corresponding to the continuous solution, that the numerical solution would, so to speak, land on the discontinuous solution and that as a result computer overflow would occur well before $t = 0$ were reached.

To explore this conjecture, we carried out a series of numerical integrations of (1.1) as initial value problems. For the continuous solution cases we integrated forwards and then backwards using the Bulirsch–Stoer method [3].[2] For the forward integration $\dot{y}(0)$ was obtained from the analytical solution (2.14). For the backward integration $y(1)$ and $\dot{y}(1)$ were obtained from the previously carried out forward integration. In addition the backward integration was carried out using the $y(1)$ and $\dot{y}(1)$ obtained from the analytical solutions (2.12) and (2.13). Table III lists for $n = 5, 6, 7, 8, 9, 10$ the initial and terminal conditions for these forward and backward integrations.[3]

From Table III we see that for $n = 9$ and 10, even with $\dot{y}(1)$ computed from the closed form solutions, backward integration resulted in overflow. We ascribe this to the inaccuracy in $\dot{y}(1)$ due, in turn, to the inaccuracy in m (see Table I) which resulted in the computation of one of the discontinuous solutions with its attendant pole.

From Figs. 2 and 3 and the closed form solution it is evident that as n increases, the solution of Troesch's problem approaches the discontinuous function which is zero for $0 \leqslant t < 1$, and unity for $t = 1$. Therefore it would appear that as n increases any numerical method of solution will experience difficulties.

The closed form solution does help to explain the success of the parameter mapping technique of Kubiček and Hlaváček [8]. By means of the transformations $w = ny$, $x = nt$, they transform (1.1), (1.2) into the problem

$$d^2w/dx^2 = \sinh w \tag{4.1}$$

$$w(0) = 0, \qquad w(n) = n. \tag{4.2}$$

For a given value of $dw(0)/dx = \dot{y}(0)$, (4.1) is integrated until the line $w = x$ is crossed. That value of x is then the value of n in (1.1), (1.2) to which the value of $\dot{y}(0)$ corresponds.

The Kubiček and Hlaváček technique solves the "inverse" problem in that it finds the n that corresponds to the correct value of $\dot{w}(0)$ rather than the other way

[2] The step size was set at $h = 0.001$ and the tolerance on the single step error for each component was set equal to 0.0001. In each component the error test tolerance is relative for those components greater than one and absolute for others (see [15]).

[3] Backward integrations using $y(1) = 1$ and $\dot{y}(1)$ from Table III were also carried out and gave essentially the same results as the BA runs in Table III.

TABLE III

Forward and Backward Integration by Bulirsch–Stoer Method

Run type[a]	n	$y(0)$	$\dot{y}(0)$	$y(1)$	$\dot{y}(1)$
F	5	0.0	0.04575046116616061	0.9999999873077304	12.10049506161658
B	5	$-6.092273666250645(10^{-11})$	0.04575046147104495	0.9999999873077304	12.10049506161658
BA	5	$-1.250679010532642(10^{-5})$	0.04581305543601977	0.9999999999994182	12.10049568741957
F	6	0.0	0.01795094997802628	1.000000090850166	20.03576338425965
B	6	$-2.427281778626515(10^{-8})$	0.01795109564192126	1.000000090850166	20.03576338425965
BA	6	$7.851220857782076(10^{-4})$	0.01323931467681633	1.000000000002039	20.03575367520284
F	7	0.0	0.00686750941624743	0.9999998080766352	33.08523302375241
B	7	$-6.501949433667376(10^{-6})$	0.006913024553472532	0.9999998080766352	33.08523302375241
BA	7	$-7.855568932826915(10^{-3})$	0.06186392587485697	0.9999999999889669	33.08526669445241
F	8	0.0	0.00258716946023976	1.000000107165949	54.5798580099281
B	8	$-2.362087662254666(10^{-3})$	0.02148419752818466	1.000000107165949	54.5798580099281
BA	8	$2.632330552197790(10^{-2})$	−0.2083969112886412	1.000000000171728	54.57982377069785
F	9	0.0	0.0009655844857949149	0.9999993698279713	90.00578403410002
B	9	overflow		0.9999993698279713	90.00578403410002
BA	9	overflow		0.9999999996559250	90.00618074169038
F	10	0.0	0.0003583377707939595	0.9999979023202755	148.405946871903
B	10	overflow		0.9999979023202755	148.405946871903
BA	10	overflow		1.000000012395226	148.4065422410549

[a] F = forward integration, B = backward integration, BA = backward integration using $y(1)$, $\dot{y}(1)$ from analytical solution.

around. The Kubiček and Hlaváček method can always find the solution for every slope, $\dot{w}(0)$, $0 < \dot{w}(0) < 1$, because the discontinuous solutions correspond to values of x greater than n. Since the curve $w(x)$ is concave upward, for a $\dot{w}(0)$ between 0 and 1 the curve will always cross the line $w = x$ at a value of x corresponding to the continuous solution before it becomes infinite at a value of x corresponding to one of the discontinuous solutions.

To illustrate this behavior let us see what happens when a value $\dot{y}(0)$ is chosen which corresponds to a discontinuous solution. For example (see Table I of Kubiček and Hlaváček [8]) $\dot{y}(0) = 0.4$ corresponds to the continuous solution for $n = 2.394$ and also to a discontinuous solution for $n \sim 9$ (see Table I). When (1.1) is integrated by the Bulirsch–Stoer method with $\dot{y}(0) = 0.4$ and $n = 2.394$, the integration proceeds smoothly, and $y(1) \approx 1$. However, when (1.1) is integrated with $\dot{y}(0) = 0.4$ and $n = 9$. overflow occurs at approximately $t = 0.334$ in agreement with our theory which locates a pole of the discontuous solution at $t_\infty =$ 0.3342. Now the integration of (4.1) with $dw(0)/dx = 0.4$ proceeds without difficulty to $x = 2.394$, thus satisfying the boundary condition $w(n) = n$. But if the integration is continued, overflow occurs at $x \approx 3$, as expected, since the pole of the discontinuous solution for $n = 9$ occurs at $x \approx 3.0074$ ($= 9 \times 0.3342$).

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